Calculation of Ultrashort Pulse Propagation based on Rational Approximations for Dispersion

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***Abstract*—Ultrashort pulses contain only a few optical cycles and exhibit broad spectra. Their carrier frequency is therefore not well defined and their description in terms of the standard slowly varying envelope approximation (SVEA) becomes questionable. Existing modeling approaches can be divided in two classes, namely generalized envelope equations, that stem from the nonlinear Schrödinger equation (NSE), and non-envelope equations which treat the field directly. Based on fundamental physical rules we will present an approach that effectively interpolates between these classes and provides a suitable setting for accurate and highly efficient numerical treatment of pulse propagation.**

I. Envelope Models

The traditional approach to the description of few-cycle optical pulses is the extension of the well-established theory of envelope equations. These models are related to the nonlinear Schrödinger equation (NSE) for the electric field envelope and several generalizations of it [1], e..g.,

$$i\partial_z\psi + \sum_{m=2}^{M_{\max}} \frac{\beta_m}{m!}(i\partial_\tau)^m\psi + \gamma|\psi|^2\psi = 0. \qquad (1)$$

After the envelope $\psi(z,\tau)$ is calculated, the electric field can be obtained from the relation

$$E(z,t) = \frac{1}{2}\psi(z, t-\beta_1 z)e^{i(\beta_0 z-\omega_0 t)} + \text{c.c.}, \qquad (2)$$

where $\tau = t - \beta_1 z$ is the retarded time.

However, the NSE, as well as its generalizations, and the very definition of the envelope (2) are based on truncated Taylor expansions of the propagation constant $\beta(\omega)$ around some reference frequency ω_0

$$\beta(\omega_0+\Omega) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!}\Omega^m. \qquad (3)$$

The principal problem with the Taylor expansion is that $\beta(\omega)$ may posses resonances, i.e., singularities for some complex values of ω, as opposed by all truncations of Eq. (3). In other words, any Taylor expansion starts to diverge when $|\Omega|$ approaches the convergence radius, and this divergence cannot be suppressed by the mechanical increase of the expansion order. This general phenomenon is illustrated in Fig. 1 for a refraction index generated by a bi-Lorentzian model for $\epsilon(\omega)$ for fluoride glass.

In particular, the Taylor expansion based generalized NSE can become invalid for ultrashort few-cycle optical pulses, whenever the pulse spectrum widens up too much. The problem becomes especially evident after consideration of the asymptotic behavior of $\beta(\omega)$, i.e., $\beta(\omega) \to \omega/c$ for $\omega \to \infty$, as opposed by behavior of the polynomial approximations.

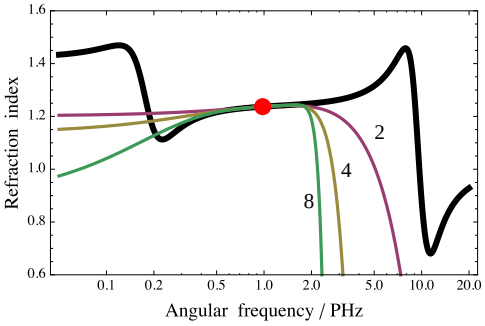

Fig. 1. Refraction index of fluoride glass (fat black line) and truncations of the Taylor expansion (color lines) of the order 2 (like in NSE), 4 and 8. Increase of the approximation order cannot resolve the divergence.

Another difficulty is related to the derivation of Eq. (1). Typically one has to assume that the evolution of the envelope $\psi(z,\tau)$ is slow as compared to $\exp i(\beta_0 z - \omega_0 t)$ factor in Eq. (2). An example is given in Fig. 2, where it is difficult to decide whether the envelope approximation is valid or not already after 10 ps of propagation time.

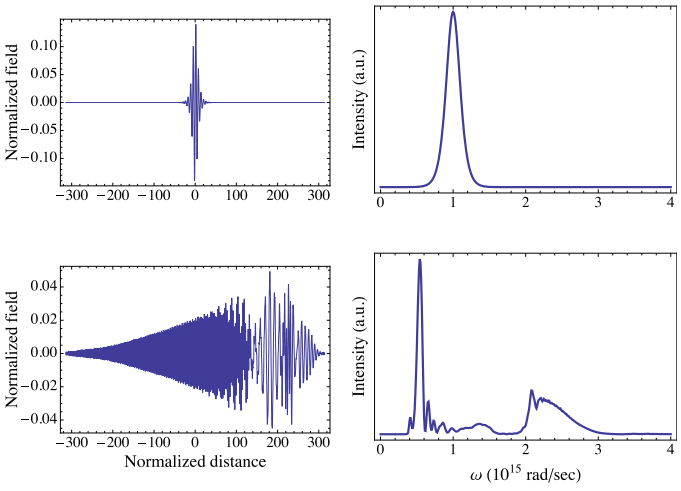

Fig. 2. Top: electric field (left) and spectrum (right) of the initial pulse. Bottom: the same after 10 ps propagation in a bulk Kerr medium with the refraction index shown in Fig. 1. The nonlinear susceptibility parameter $\chi^{(3)}$ is used to normalize the fields.

II. Short Pulse Equations

The alternative approach is to treat the electric field directly, guiding into the class of short-pulse equations (SPE) [2]–[4]. Typically such models assume a reasonably simplified model for the medium dispersion and an instantaneous cubic nonlinearity. As an example, the dispersion relation with two fit parameters $\epsilon(\omega) = \bar{\epsilon}(1 - \mu^2/\omega^2)$ for the refractive index in the transparency window (see Fig. 3) guides to the SPE for the properly scaled electric field $U(z,\tau)$ [2], [3]:

$$\partial_z U + \int_{-\infty}^{\tau} U(z,\tau')d\tau' + U^2\partial_\tau U = 0, \qquad (4)$$

where also exact soliton solutions have been found [5]. A more general rational fit $\epsilon(\omega) = \bar{\epsilon}\left(1 - \mu^2/\omega^2 + \nu^2\omega^2\right)$ is especially useful near the zero dispersion wavelength, where interesting effects by the transition between anomalous and normal dispersion are expected [6]. Here, the only possible solitary solution is a solitary breather that exhibits some intrinsic dynamics [3], [5], [7].

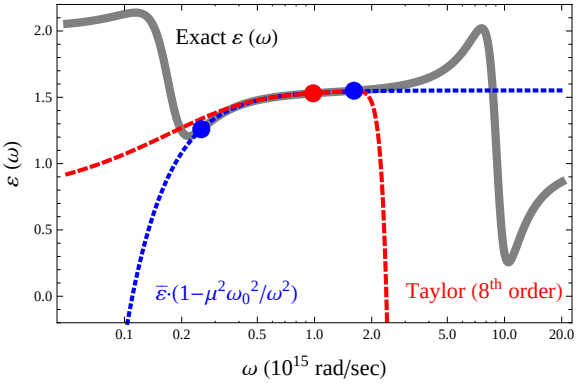

Fig. 3. Material response dispersion of fluoride glass (fat grey line) and a simple rational approximation to it (blue dashed line). For comparison also Taylor expansion of 8th order (red dashed line) is shown [3].

III. Rational Approximation

More generally, following the alternative approach, we employ a rational fit for the material dispersion, e.g. [8]:

$$n(\omega_0 + \Omega) \approx \frac{p_0 + p_1\Omega + \cdots + p_M\Omega^M}{1 + q_1\Omega + \cdots + q_N\Omega^N}. \qquad (5)$$

A principle advantage of Eq. (5) over Eq. (3) is that it accounts for complex singularities and approximates $n(\omega)$ in a considerably larger frequency domain, c.f. Fig. 4.

Moreover, by choosing polynomials of the same power $M = N$ in the numerator and denominator of Eq. (5) one additionally ensures that $n(\omega)$ remains bounded for $\omega \to \infty$, as it physically has to be, and as it is opposed by unbounded polynomial approximations. Even more, by proper locating the poles in the complex frequency domain, one can ensure causality, as a basic physical demand. Among other things, this leads to a considerable reduction of the numerical stiffness when computing solutions of the envelope equations. As a consequence, the resulting propagation equations become nonlocal in time which expresses the delayed medium response. Structurally, the propagation equation becomes:

$$i\partial_z\Psi + (\omega_0 + i\partial_t)\left[\frac{1}{c}\hat{\mathfrak{D}}_n\Psi + \tilde{\gamma}\,\hat{\mathfrak{D}}_n^{-1}|\Psi|^2\Psi\right] = 0 \qquad (6)$$

where the nonlocal operators $\hat{\mathfrak{D}}_n$, $\hat{\mathfrak{D}}_n^{-1}$ are defined as convolutions with the dispersion function (5).

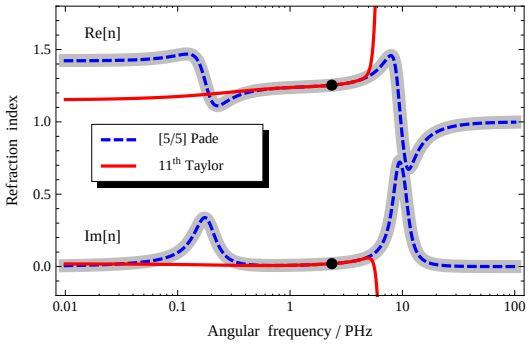

Fig. 4. Complex dispersion for the same material as in Fig. 1 (fat grey). Red: Taylor expansion (11th order), blue: [5/5] Pade approximation [8].

IV. Conclusion

We use the Pade approximation indicated in Fig. 4 to quantify the general dispersion operator in Eq. (6). The latter model is similar to non-envelope equations as it can be derived without the slowly-varying envelope approximation. On the other hand, Eq. (6) possesses the relatively simple structure of the generalized NSE and can be effectively addressed by adaptation of the existing numerical solvers.

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References

[1] A. Demircan and U. Bandelow, "Analysis of the interplay between soliton fission and modulation instability in supercontinuum generation," *Appl. Phys. B*, vol. 86, no. 1, pp. 31–39, 2007.

[2] S. A. Kozlov and S. V. Sazonov, "Nonlinear propagation of optical pulses of a few oscillations duration in dielectric media," *JETP*, vol. 84, no. 2, pp. 221–228, 1997.

[3] S. Amiranashvili, A. G. Vladimirov, and U. Bandelow, "Solitary wave solutions for few-cycle optical pulses," *Phys. Rev. A*, vol. 77, p. 063821, 2008.

[4] M. E. Pietrzyk, I. Kanattšikov, and U. Bandelow, "On the propagation of vector ultra-short pulses," *J. Nonlin. Mathematical Physics*, vol. 15, no. 2, pp. 162–170, 2008.

[5] A. Sakovich and S. Sakovich, "The short pulse equation is integrable," *J. Phys. Soc. Jpn.*, vol. 74, no. 1, pp. 239–241, 2005.

[6] S. Amiranashvili, A. G. Vladimirov, and U. Bandelow, "A model equation for ultrashort optical pulses around the zero dispersion frequency," *Europ. Phys. J. D*, vol. 58, pp. 219–226, 2010.

[7] S. A. Skobelev, D. V. Kartashov, and A. V. Kim, "Few-optical-cycle solitons and pulse self-compression in a Kerr medium," *Phys. Rev. Lett.*, vol. 99, no. 20, p. 203902, 2007.

[8] S. Amiranashvili, U. Bandelow, and A. Mielke, "Pade approximant for refractive index and nonlocal envelope equations," *Opt. Commun.*, vol. 283, pp. 480–485, 2010.