Bifurcation analysis of traveling wave models

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***Abstract*—We present a general method allowing to perform a direct bifurcation diagram and to determine the stability of the solutions of a Traveling wave model. We exemplify the cases of the two level atom and the semiconductor quantum well lasers.**

I. Introduction

The Maxwell-Bloch equations governing the laser dynamics can be regarded as a singularly perturbed problem: the fast relaxation of the different polarization components is responsible of the wide gain bandwidth which in turn induces a very weak discrimination between the laser modes. These weak modal gain differences are what ultimately govern the evolution of the field and a completely different scenario may be expected if these small terms are not properly accounted for.

Traveling Wave models (TWM) has proven to be a very useful tool for the study of problems involving broadband multimode dynamics, as for instance the directional switching in ring lasers induced by short pulses injection [1] or the Mode-Locking of Fabry-Pérot lasers with intracavity saturable absorbers [2]. Although not trivial a problem, the direct time integration of TWMs is now well understood. It is however a limited tool if one wants to reconstruct a bifurcation diagram.

It is possible to circumvent the direct study of the Maxwell-Bloch equations via a spatially resolved TWM by invoking the Uniform Field Limit (UFL), where one assume small gain and losses and an almost conservative cavity. In this case, a modal decomposition can be sought resulting in a low number of coupled ordinary differential equations (ODE)s that can be studied, via standard bifurcation methods. Not surprisingly, only a handful of results are known out of the UFL. In addition, since the UFL is robust a limit, one may be tempted to extend results out of their domain of validity. There are cases where this approach is incorrect; a prominent example is the disappearance of the Risken-Nummedal-Graham-Haken instability when the mirror losses are increased [5]. This qualitative change renders difficult to infer the real domain of validity of the UFL. It is important to notice that the UFL usually does not apply to semiconductor lasers.

Software packages like AUTO [3] or DDEbifTool [4] allow to perform a numerical bifurcation analysis of systems composed of a moderate number of coupled ODEs or delayed differential equations (DDE)s. By mapping the various steady and oscillating regimes encountered when one –or several– control parameters are varied and by determining their linear stability analysis (LSA) one can elaborate a global dynamical scenario. This approach is the most efficient one to assess the performance of a device. For instance, in control theory being able to calculate the rightmost, stability determining roots of the LSA problem gives a much better prediction of the asymptotic stability than what could be obtained by direct numerical time integration. The LSA does not suffer from the critical slowing down phenomenon near a bifurcation.

Although there are no equivalent softwares adapted to the bifurcation analysis of partial differential equations (PDE)s, a relatively coarse discretization of the variables allows to approximate the spatial operators via finite differences. In this way, one can recast an PDE into a large ensemble of sparsely coupled ODEs. This transformation usually induces numerical dissipation which may not pose a problem if the underlying dynamical system is already dissipative. As such, this method is adapted to the study of shallow wave solutions of parabolic PDEs, like e.g. the so-called Brusselator model. However, this approach completely fails to describe the evolution of an almost conservative hyperbolic PDE.

In this paper we present numerical methods – independent of the boundary conditions, of the parameters values and whether or not the UFL applies – that allows for a direct bifurcation analysis of a spatially resolved TWM.

II. Results

Our model, in the case of a homogeneously broadened lasers (see [6] and reference therein) reads

$$\left(\partial_\tau \pm \partial_s\right) A_\pm = B_\pm - \alpha A_\pm, \quad (1)$$

$$\gamma^{-1}\partial_\tau B_\pm = -\left(1+i\delta\right) B_\pm + D_0 A_\pm + D_{\pm 2} A_\mp, \quad (2)$$

$$\epsilon^{-1}\partial_\tau D_0 = J - D_0 - 2\Re\left(A_+ B_+^* + A_- B_-^*\right), \quad (3)$$

$$\eta^{-1}\partial_\tau D_{\pm 2} = -D_{\pm 2} - \epsilon\eta^{-1}\left(A_\pm B_\mp^* + A_\mp^* B_\pm\right), \quad (4)$$

where $A_\pm$ are the scaled slowly varying amplitudes of the counter-propagating electric fields, $B_\pm$ are their respective polarizations, D_0 is the quasi-homogeneous inversion density and $D_{\pm 2}$ are the spatially-dependent contributions to the grating in the population inversion density that arise from standing wave effects and lead to saturation of the gain. Space and time (s, τ) are scaled by the length and the time of flight of the cavity, respectively. α are the internal losses per pass, γ determines the spectral width of the gain spectrum, δ is the detuning between the atomic resonance and the nearest cavity mode, and ϵ and η are the decay times for D_0 and $D_{\pm 2}$ respectively, which differ due to the impact of diffusion on

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the decay of the grating terms. We treat on equal grounds ring and Fabry-Pérot cavities by supplying for general boundary conditions that reads

$$A_+(0,\tau) = t_+A_+(1,\tau) + r_-A_-(0,\tau), \quad (5)$$
$$A_-(1,\tau) = t_-A_-(0,\tau) + r_+A_+(1,\tau). \quad (6)$$

We determine the monochromatic solutions of (1-4) for a given cavity configuration via a low dimensional shooting method. With an initial guess for the modal frequency and amplitudes $A_\pm(0)$, we solve for the spatial dependence of (1)-(4) towards the other end of the cavity, where the propagated values $A_\pm(1)$ must verify the boundary conditions. A Newton-Raphson algorithm provides a new guess for the field amplitudes $A_\pm(0)$ and the modal frequency and the process is repeated until one reaches convergence. The final trajectory generated by this shooting method provides a discretized representation over a mesh of N samples of the modal profile.

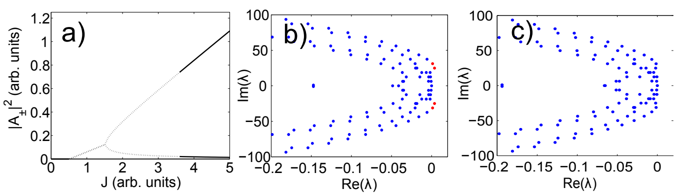

Fig. 1. (a) Numerical bifurcation diagram for mode $m = 2$ for a ring laser, $\gamma = 250$, $\alpha = 2.03$, $\epsilon = 0.05$, $\eta = 10$, $g = 4$, $t_\pm = 0.98$ and $r_\pm = 0.01$. Eigenvalue spectra for $J = 3$ (b) and $J = 4$ (c).

Once the monochromatic solutions are determined, one could in principle compute the eigenvalues from (1)-(4), linearized around the newly found solution. However, the resulting system is still a hyperbolic PDE, and a discrete representation of the solution would require to express the gradient operator using finite differences. This approach is not practical: time propagation of hyperbolic PDEs cannot be made reliably for an arbitrary choice of the spatial and of the temporal discretization, leading to large errors in the eigenvalues. Instead, we use the temporal map $\mathbf{V}_{n+1} = \mathbf{U}(h, \mathbf{V}_n)$ described in [6] that advances the state vector $\mathbf{V}$ a time step $h = 1/N$ while verifying the Courant condition and canceling numerical dissipation. We then consider all possible perturbations of $\mathbf{V}$ hereby finding the matrix $\mathbf{M} = \partial\mathbf{U}/\partial\mathbf{V}$ representing the linear operator governing the time evolution for the perturbations around one mode. We finally compute the $11 \times N$ Floquet multipliers z_i of $\mathbf{M}$ by a standard QR decomposition and determine the eigenvalues as $\lambda_i = h^{-1}\ln z_i$.

The results of this procedure are shown in Fig. 1 for one of the side mode of a symmetric, bidirectional ring laser with a point coupler. This is a pathologic case since both $t_\pm$ and $r_\pm$ are different from zero. However, it represents accurately semiconductor ring laser [7]. Solid (dashed) lines represent the stable (unstable) solutions. In panel a) we show that just above the threshold current $J \simeq 0.51$, the solution corresponds to an unstable bidirectional state. At $J \simeq 1$, a pitchfork bifurcation into unidirectional emission occurs, but the degenerate (almost) unidirectional states are also unstable, as evidenced by the eigenvalues shown in panel b) for $J = 2$. However, for currents above $J > 2.3$, they become stable and all the eigenvalues have $\Re(\lambda) < 0$ (see panel c) for $J = 3$). Our method allowed us to discuss the impact of the cavity configuration on the possible longitudinal mode multistability in ring lasers [8] and its use in all optical signal processing.

To conclude, while the model presented in eqs. (1)-(4) is adapted to the case of solid state and CO_2 lasers, it must be modified to describe the optical response of semiconductor Quantum-Well (QW). We do so by using the method recently presented in [9] where we evaluate the time-domain susceptibility of the QW by an integral kernel. By using a similar methodology, one may obtain an equivalent bifurcation diagram. For instance, we represented in Fig. 2 the eigenvalue spectrum of the off solution of a Fabry-Pérot QW laser with cleaved facets. One can clearly recognize the blue shifted, asymmetrical shape of gain and the presence of the band-edge.

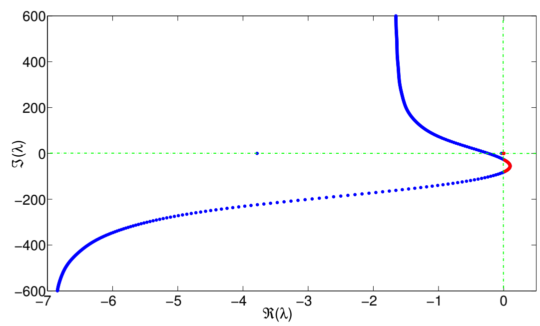

Fig. 2. Eigenvalue spectrum of the off solution of a Fabry-Pérot QW Laser.

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