On the compression of ultrashort optical pulses beyond the slowly varying envelope approximation

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Outlook

- Physical motivation
- Derivation of generalized Short Pulse Equation
- Generalized SPE $\rightarrow$ generalized NSE
- SPE and its properties (conservation law, soliton solutions)
- A multisymplectic integrator for SPE
- Pulse compression
- Conclusions

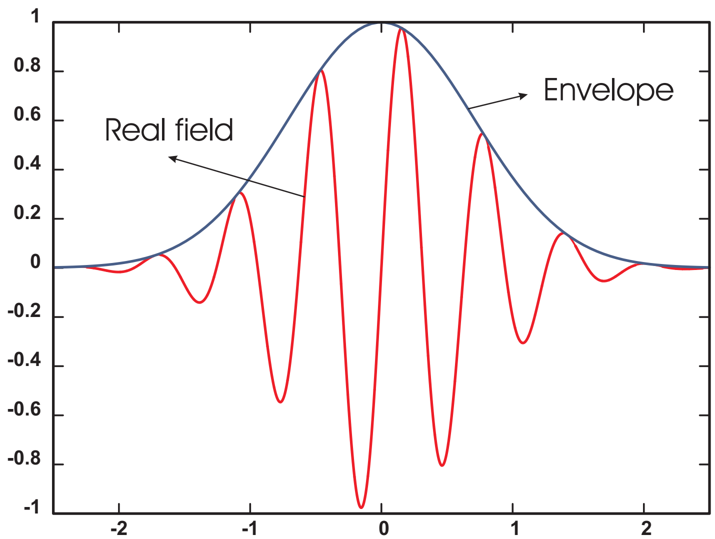
Physical motivation

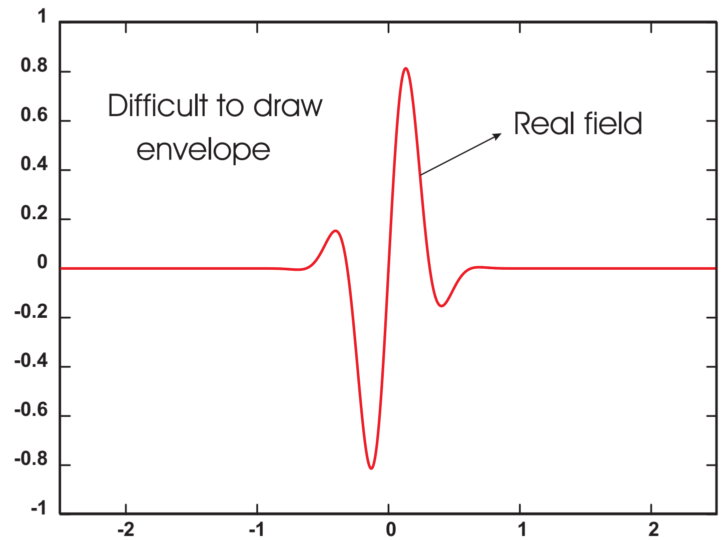
NSE: $i\Psi_z = \Psi_{tt} + |\Psi|^2\Psi$

equation for an envelope of optical field, many-cycle ($\sim 100 fs$) pulses

SPE: $u_{zt} = u + (u^3)_{tt}$

equation for real optical filed, ultra-short, few-cycle ($\sim fs$) pulses





- Propagation of optical pulses in Kerr media is usually described by the NSE (an equation for the envelope of the optical field), derived using the slowly varying envelope approximation (SVEA).
- The notion of an envelope of ultrashort pulses whose temporal extend is less than a few cycles of the corresponding wave is doubtful.
- SPE derived by Schäfer and Wayne, Phys. D **196**, 90 (2004) using multiplescale method.

Derivation of generalized Short Pulse Equation (1)

- Maxwell equations for a medium free of charges and currents:

$$\nabla \times \mathbf{E}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) = 0$$
$$\nabla \times \mathbf{H}(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) = 0$$
$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$$

- constitutive relations for a non-magnetic medium:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu_0 \mathbf{H}$$

$\mathbf{E}$ ($\mathbf{H}$) - electric (magnetic) fields
$\mathbf{D}$ ($\mathbf{B}$) - electric and magnetic inductions
$\varepsilon_0(\mu_0)$ - vacuum permittivity (permeability)

Derivation of generalized Short Pulse Equation (2)

- induced electric polarization - sum of linear and nonlinear terms, $\mathbf{P} = \mathbf{P}^L + \mathbf{P}^{NL}$
- isotropic (second-order nonlinear polarization vanishes) and piecewide uniform medium, $\chi_{ij} \neq \chi_{ij}(\mathbf{r})$ and $\chi_{ijkl} \neq \chi_{ijkl}(\mathbf{r})$

$$P_i^L(\mathbf{r}, t) = \varepsilon_0 \int \chi_{ij}(t - \bar{t}) E_j(\mathbf{r}, \bar{t})\, d\bar{t},$$

$$P_i^{NL} = \varepsilon_0 \int \chi_{ijkl}(t - \bar{t}_1, t - \bar{t}_2, t - \bar{t}_3) E_j(\mathbf{r}, \bar{t}_1) E_k(\mathbf{r}, \bar{t}_2) E_l(\mathbf{r}, \bar{t}_3)\, d\bar{t}_1\, d\bar{t}_2\, d\bar{t}_3.$$

Using the following facts

- $\nabla \cdot \mathbf{E} = 0$ (it follows from the Maxwell equations in the case of a uniform medium)
- $\nabla \times \nabla \times \mathbf{E} \equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$

one gets the following wave equation:

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{D} \tag{1}$$

Derivation of generalized Short Pulse Equation (3)

Let us consider the following type of an optical fiber:

- single mode fiber, $V < V_{cr} \approx 2.405$
- weakly guiding, $\Delta << 1$
- step-index profiled (as an example)

In such a case we can represent the electric field in the following separable form:

$$\mathbf{E}(\mathbf{r},\mathbf{t}) = \hat{\mathbf{x}}\, F(x,y)\, \Psi(z,t)$$

where
$F(x,y)$ - fundamental mode field
$V \equiv \frac{\omega}{c}\, \rho\, n_{core}\, \sqrt{2\Delta}$ - fiber parameter
$\Delta \equiv \frac{n_{core}^2 - n_{clad}^2}{2n_{core}^2}$ - profile high parameter
$n_{core}(n_{clad})$ - refractive index of the fiber core (cladding)
ρ - fiber radius

an example:
$n_{core} = 1.445, \;\; n_{clad} = 1.45 \quad \rightarrow \Delta \approx 3 \times 10^{-3}$
$\rho = 3\, \mu m, \;\; V < V_{cr} \approx 2.405 \;\; \rightarrow \lambda > 0.945\, \mu m$

Derivation of generalized Short Pulse Equation (4)

- assuming that the nonlinear response of the medium is instantaneous:

$$\chi_{1111}(t-\bar{t}_1, t-\bar{t}_2, t-\bar{t}_3) = \chi^3\,\delta(t-\bar{t}_1)\,\delta(t-\bar{t}_2)\,\delta(t-\bar{t}_3)$$

we get:

$$\tilde{\mathbf{D}}(\mathbf{r},\omega) = \hat{\mathbf{x}}\,\varepsilon_0\,(1+\tilde{\chi}(\omega))\,F(x,y)\,\tilde{\Psi}(z,\omega) + \hat{\mathbf{x}}\,\varepsilon_0\,\chi^3\left(F(x,y)\,\tilde{\Psi}(z,\omega)\right)^3$$

where $\tilde{\chi}^{(1)}(\omega) \equiv \int \chi_{ij}(t)e^{(i\omega t)}dt$ and $\tilde{\Psi}(z,\omega) \equiv \int \Psi(z,t)e^{(i\omega t)}dt$ are Fourier tarnsforms

- using the method of separation of variables (applicable in the case of small nonlinearity) the wave equation can be written as the following system of equations:

$$F(x,y)\frac{\partial^2}{\partial z^2}\tilde{\Psi}(z,\omega) = -F(x,y)\,\beta^2(\omega)\,\tilde{\Psi}(z,\omega) - F^3(x,y)\,\chi^3\,\frac{\omega^2}{c^2}\tilde{\Psi}^3(z,\omega) \qquad (2)$$

$$\Delta_\perp F(x,y) = \left(\beta^2(\omega) - \frac{\omega^2}{c^2}\,(1+\chi(\omega))\right)F(x,y) \qquad (3)$$

- in Eq. (3) (wave equation for mode $F(x,y)$) nonlinearity has been ignored

- integrating Eq. (2) over the transverse cross section we get:

$$\frac{\partial^2}{\partial z^2}\,\tilde{\Psi}(z,\omega) = -\chi^3\beta^2(\omega)\,\tilde{\Psi}(z,\omega) - \frac{\omega^2}{c^2}\,\frac{\iint F^4(x,y)dxdy}{\iint F^2(x,y)dxdy}\,\tilde{\Psi}^3(z,\omega) \qquad (4)$$

Derivation of generalized Short Pulse Equation (5)

- propagation constant for step-index fiber: $\beta(\omega) = \frac{1}{\rho}\left(\frac{V^2}{2\Delta} - U^2\right)$
- U - core parameter, within Gaussian approximation $U \approx \sqrt{1 + 2\ln V} < V$
- weakly guiding fibers: $\Delta << 1$

Contribution of the waveguiding effect to the propagation constant $\beta(\omega)$ is therefore small and can be neglected,

$$\beta^2(\omega) \approx \frac{\omega^2}{c^2}(1 + \tilde{\chi}(\omega)) = \frac{\omega^2}{c^2} n_L^2(\omega) \tag{5}$$

- square electric field $\Psi^2[1/V^2] \to$ pulse power $P_0\, U^2\, [W/m^2]$

$$\chi^3\Psi^2(z,\omega) = \chi^3\Psi_0 U^2(z,\omega) = \frac{8}{3} n_0 n_2 I_0\, U^2(z,\omega) = \frac{8}{3} n_0 n_2 \frac{P_0\, U^2(z,\omega)}{\iint F^2(x,y)dxdy} \tag{6}$$

$n_0 = n_L(\omega_0)$ - linear refractive index,
$n_2 : n = n_0 + n_2 I$ - second order (Kerr) nonlinear refractive index,
ω_0 - central frequency of the pulse,
$I_0\, [W]$ - pulse peak intensity,
$P_0\, [W/m^2]$ - pulse peak power.

Derivation of generalized Short Pulse Equation (6)

- Linear refractive index for the transparency range can be well approximated by the Sellmeier formula: ($\rightarrow$ Lorentz model of the medium with neglected absorption)

$$n_L^2(\lambda) = 1 + \chi(\lambda) = 1 + \sum_{i=1}^{3} \frac{B_i \lambda^2}{\lambda^2 - \lambda_i^2}$$

for bulk silica: $B_1 = 0.6961663,\ B_2 = 0.4079426,\ B_3 = 0.8974794$
$\lambda_1 = 0.0684043\,\mu m, \lambda_2 = 0.1162414\,\mu m, \lambda_3 = 9.896161\,\mu m.$

- Assuming that the frequency range of the pulse is far away from any resonance of the medium, $\lambda_1 < \lambda_2 << \lambda << \lambda_3$ one can simplify the Sellmeier formula:

$$\chi(\lambda) \approx \chi_0 \ + \ \chi_2\,\lambda^2 \ + \ \chi_4\,\lambda^4 \ + ... + \ \chi_{-2}/\lambda^2 \ + \ \chi_{-4}/\lambda^4 \ + ...$$

- in the range: $1.5\mu m - 3.0\mu m$ (anomalous dispersion) we can use approximation: $\chi(\lambda) \approx 1.1104 - 0.01063\,\lambda^2$ (error of accuracy 5×10^{-4})

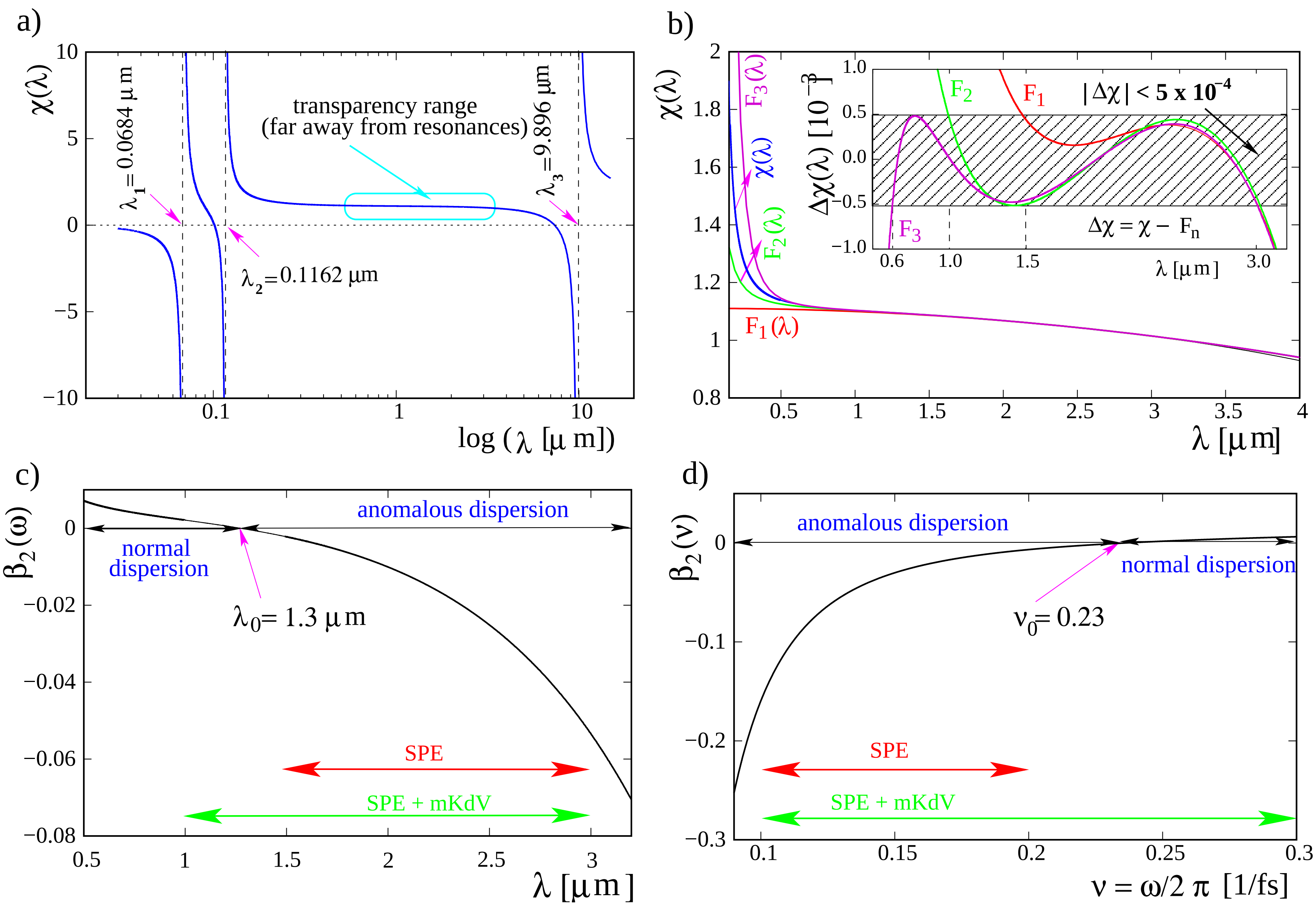
- in the range: $1.0\mu m - 3.0\mu m$ (normal and anomalous dispersion) we can use approximation $\chi(\lambda) \approx 1.1109 - 0.01054\,\lambda^2 + 0.0048/\lambda^2$ (error of accuracy 5×10^{-4})

thus

$$n_L^2 = 1 + \tilde{\chi}(\omega) = 1 + \chi_0 + \bar{\chi}_2\omega^2 - \bar{\chi}_{-2}/\omega^2 = N_0^2 - 2N_0\,c\,a_1/\omega^2 + 2N_0\,c\,a_2\,\omega^2, \qquad (7)$$

where $N_0 = \sqrt{1+\chi_0}$, $a_1 = \frac{\bar{\chi}_{-2}}{2N_0\,c}$ and $a_2 = \frac{\bar{\chi}_2}{2N_0\,c}$

Derivation of generalized Short Pulse Equation (7)



Derivation of generalized Short Pulse Equation (8)

Applying Fourier transform to Eq. (4) and taking into account formulas (5)- (8) we obtain

$$\frac{\partial^2}{\partial z^2}U(z,t) = \frac{N_0^2}{c^2}\frac{\partial^2}{\partial t^2}U(z,t) + \frac{2N_0 a_1}{c}U(z,t) - \frac{2N_0 a_2}{c}\frac{\partial^4}{\partial t^4}U(z,t) + \frac{2n_0 n_2}{c^2 A_{eff}}P_0\frac{\partial^2}{\partial t^2}U^3(z,t)$$

where $A_{eff} = \frac{\left(\iint F^2(x,y)dxdy\right)^2}{\iint F^4(x,y)dxdy}$ is effective pulse area

Applying

- Galilean transformation: $\tau = t - \frac{N_0}{c}z$ $\quad (\frac{\partial^2}{\partial z^2} \to \frac{N_0^2}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2N_0}{c}\frac{\partial^2}{\partial z \partial t} + \frac{\partial^2}{\partial z^2},\ \frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau})$
- unidirectional propagation, $\frac{\partial^2}{\partial z^2} = 0$

we get the generalized Short Pulse Equation:

$$\frac{\partial^2}{\partial z \partial \tau}U(z,\tau) = a_1\, U(z,\tau) - a_2\frac{\partial^4}{\partial \tau^4}U(z,\tau) + g\frac{\partial^2}{\partial \tau^2}U^3(z,\tau), \qquad (8)$$

where $a_1 = \frac{\bar{\chi}_2}{2N_0 c}$, $a_2 = \frac{\bar{\chi}_{-2}}{2N_0 c}$, and $g = \frac{n_0 n_2}{A_{eff}}\frac{P_0}{N_0 c}$

generalized SPE → generalized NSE

$$\frac{\partial^2}{\partial z \partial \tau} U(z,\tau) = a_1 U(z,\tau) - a_2 \frac{\partial^4}{\partial \tau^4} U(z,\tau) + g \frac{\partial^2}{\partial \tau^2} U^3(z,\tau),$$

- Ansatz: $U = \frac{1}{2}\left(\mathcal{E} e^{-i(\bar{k}_0 z + \omega_0 t)} + c.c.\right)$, ω_0 - any frequency
- Using: $k(\omega) = a_2 \omega^3 - \frac{a_1}{\omega}$; $\beta_k := \frac{1}{k!} \frac{\partial^k k(\omega)}{\partial \omega^k}\Big|_{\omega_0}$; $k = 1, 2, \ldots$; $\beta_0 = k(\omega_0) - \beta_1 \omega_0$
- Defining: $T = \tau + \beta_1 z$ and keeping only terms $\sim e^{-i\omega_0 T}$ one gets:

$$\frac{\partial \mathcal{E}}{\partial z} = \sum_{k=2}^{\infty} i^{k-1} \beta_k \frac{\partial^k \mathcal{E}}{\partial T^k} - i\gamma_1 |\mathcal{E}|^2 \mathcal{E} + \gamma_2 \frac{\partial}{\partial T}\left(|\mathcal{E}|^2 \mathcal{E}\right)$$
$$+\gamma_2 \left(\mathcal{E}^2 \frac{\partial \mathcal{E}}{\partial T} - i\omega_0 \mathcal{E}^3\right) e^{-2\left(\bar{\beta}_0 z + \omega_0 T\right)}$$

which represents the standard NSE with additional terms (infinite number of dispersive terms, self-steepening and third-harmonic generation)

- J. Dudly → NSE with third-harmonic generation term

SPE and its properties

generalized SPE:

$$\frac{\partial^2}{\partial z \partial \tau} U(z,\tau) = a_1\, U(z,\tau) - a_2 \frac{\partial^4}{\partial \tau^4} U(z,\tau) + g \frac{\partial^2}{\partial \tau^2} U^3(z,\tau),$$

$a_1 = 0$ (normal dispersion regime) $\rightarrow$ mKdV:

$$\frac{\partial^2}{\partial z \partial \tau} U(z,\tau) = -a_2 \frac{\partial^4}{\partial \tau^4} U(z,\tau) + g \frac{\partial^2}{\partial \tau^2} U^3(z,\tau),$$

$a_2 = 0$ (anomalous dispersion regime) $\rightarrow$ SPE:

$$\frac{\partial^2}{\partial z \partial \tau} U(z,\tau) = a_1\, U(z,\tau) + g \frac{\partial^2}{\partial \tau^2} U^3(z,\tau),$$

mKdV and SPE are integrable equations

Integrability of the SPE proven independently by:

- Sakovich & Sakovich $\rightarrow$ zero curvature representation (J. Phys. Soc. Jpn. **74** (2005) 239)
- Brunelli $\rightarrow$ bi-hamiltonian system (Phys. Lett. A **353** 475-478 (2006))

- $\frac{\partial}{\partial z} \int_{-\infty}^{\infty} U^2\, d\bar{\tau} = 0$ (conservation of energy)
- $\int_{-\infty}^{\infty} U\, d\bar{\tau} = 0 \rightarrow \hat{U}(z, \omega = 0) = 0$
  (no zero-frequency component - a property of optical fields)

The breather solution of SPE

- Sakovich & Sakovich and Brunelli have shown that SPE is integrable
- Sakovich & Sakovich: the SPE transforms to the sine-Gordon equation by a chain of transformations. When applied to the breather solution of the sine-Gordon equation:

$$\phi = -4\arctan\left(\frac{m\sin(\psi)}{n\cosh(\theta)}\right)$$

it gives a breather solutions of SPE:

$$u(\tau) = 4mn\frac{m\sin(\psi)\sinh(\theta) + n\cos(\psi)\cosh(\theta)}{m^2\sin^2(\psi) + n^2\cosh^2(\theta)},$$

where

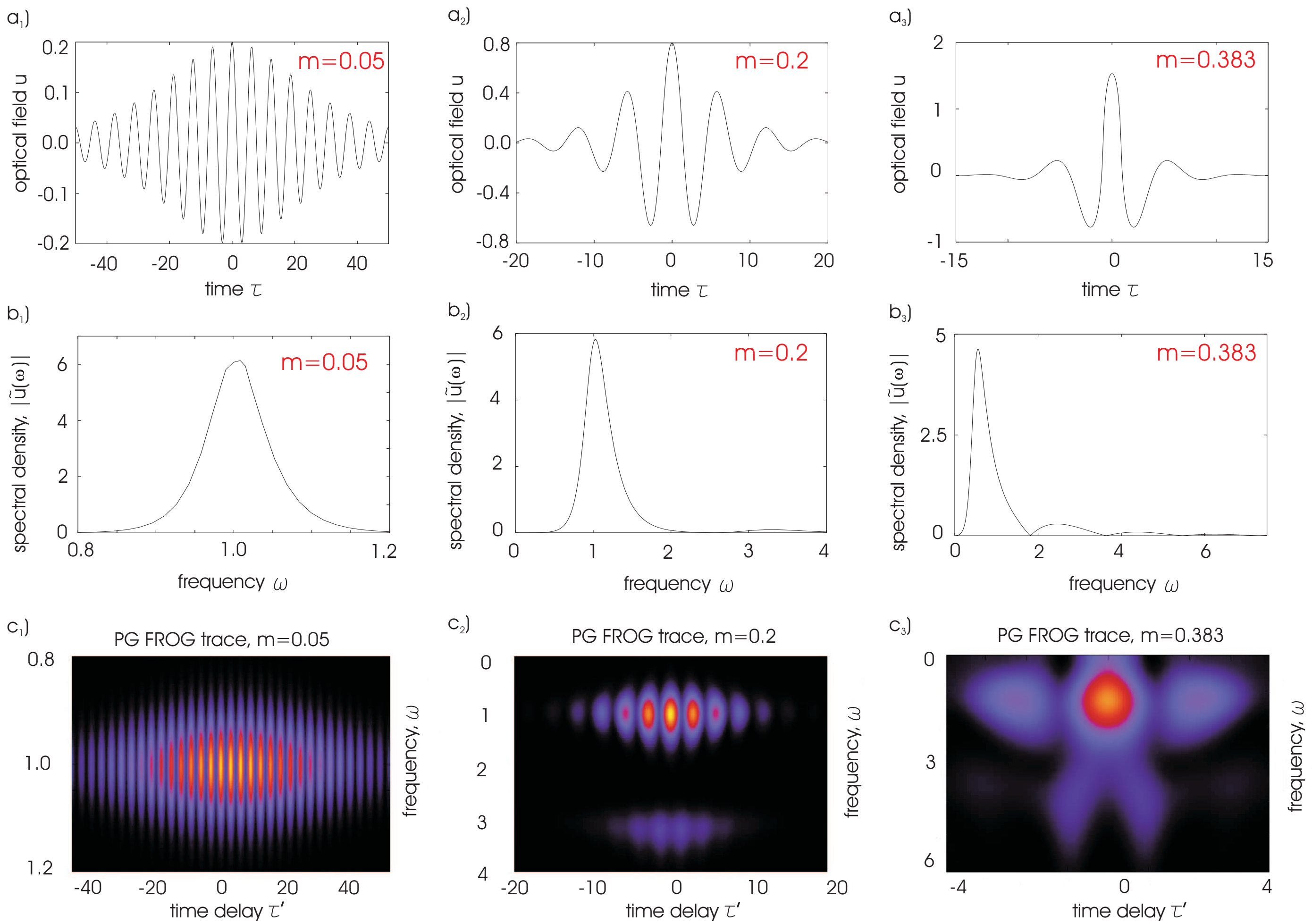
$\tau = t + 2mn\frac{m\sin(2\psi) - n\sinh(2\theta)}{m^2\sin^2(\psi) + n^2\cosh^2(\theta)}$,

$\theta = m(t + z)$,

$\psi = n(t - z)$,

$n = \sqrt{1 - m^2}$.

The breather solution of SPE



- small m - long pulses and small amplitude E_s, for $m \to 0$ soliton envelope $\to$ sech-shape, $U \approx 4m\, cos(t - z)\, sech(m(t = z))$
- short pulses - broad spectrum with higher-harmonics peaks
- $m > m_{crit} \approx 0.383$ - singular solution (non single-valued)

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{9}$$

can be represented in the form

$$\phi_{xt} - \phi - \frac{1}{6}(\phi_x^3)_x = 0 \tag{10}$$

if one introduces the potential

$$u = \phi_x. \tag{11}$$

This equation can be derived from the first order Lagrangian

$$L = \frac{1}{2}\phi_t\phi_x - \frac{1}{24}\phi_x^4 + \frac{1}{2}\phi^2. \tag{12}$$

The multisymplectic formulation of SPE (2)

Using the standard prescription of the multisymplectic (De Donder-Weyl) Hamiltonian formalism, we introduce the polymomenta

$$\begin{aligned} p^t &:= \frac{\partial L}{\partial \phi_t} = \frac{1}{2}\phi_x \\ p^x &:= \frac{\partial L}{\partial \phi_x} = \frac{1}{2}\phi_t - \frac{1}{6}\phi_x^3 \end{aligned} \qquad (13)$$

and the (De Donder-Weyl) Hamitlonian

$$H_{DW} := p^t\phi_t + p^x\phi_x - L = 2p^x p^t + \frac{2}{3}(p^t)^4 - \frac{1}{2}\phi^2. \qquad (14)$$

Then the multisymplectic (De Donder-Weyl) Hamiltonian equations take the form:

$$\begin{aligned} \partial_x p^x + \partial_t p^t &= -\frac{\partial H}{\partial \phi} = \phi \\ \partial_x \phi &= \frac{\partial H}{\partial p^x} = 2p^t \\ \partial_t \phi &= \frac{\partial H}{\partial p^t} = 2p^x + \frac{8}{3}(p^t)^3. \end{aligned}$$

It is obtained from the discretization of DW Hamiltonian equations using the midpoint method in both x and t directions:

$$\begin{aligned}
&\phi_{i,j} \approx \phi(i\Delta x, j\Delta t), \\
&\phi_{i+1,j+1/2} := \frac{1}{2}(\phi_{i,j} + \phi_{i,j+1}) \\
&\phi_{i+1/2,j+1/2} := \frac{1}{4}(\phi_{i,j} + \phi_{i,j+1} + \phi_{i+1,j} + \phi_{i+1,j+1})
\end{aligned} \tag{15}$$

We obtain:

$$\frac{p^x_{i+1,j+\frac{1}{2}} - p^x_{i,j+\frac{1}{2}}}{\Delta x} + \frac{p^t_{i+\frac{1}{2},j+1} - p^t_{i+\frac{1}{2},j}}{\Delta t} = \phi_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$\frac{\phi_{i+1,j+\frac{1}{2}} - \phi_{i,j+\frac{1}{2}}}{\Delta x} = 2p^t_{i+\frac{1}{2},j+\frac{1}{2}} \tag{16}$$

$$\frac{\phi_{i+\frac{1}{2},j+1} - \phi_{i+\frac{1}{2},j}}{\Delta t} = 2p^x_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{8}{3}(p^t_{i+\frac{1}{2},j+\frac{1}{2}})^3 \tag{17}$$

THEOREM: *The above integrator fulfils the discretized multisymplectic conservation law.*

A multisymplectic integrator for SPE

By a tedious but straightforward calculation we obtain:

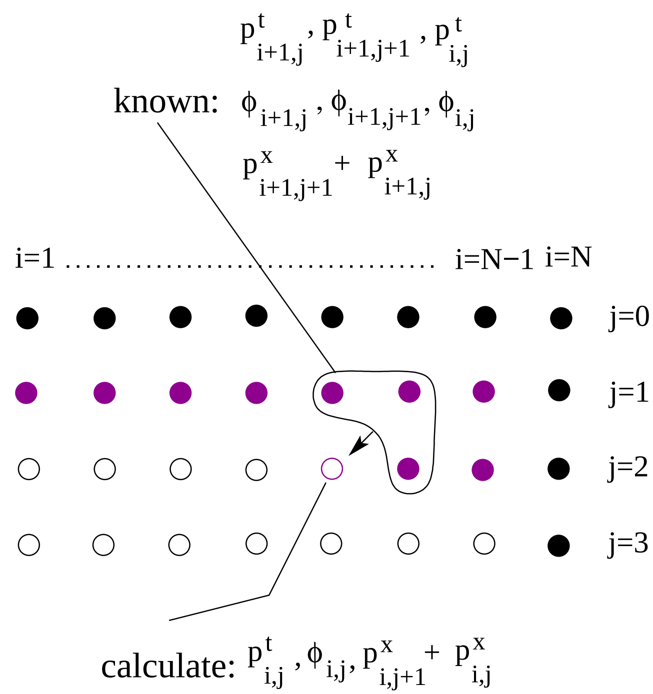
$$(p^t_{i+1,j+1})^3+3p^t(i{+}{+})(p^t_{i+1,j+1})^2+3\left((p^t(i++))^2-4\Delta_{xt}\right)(p^t_{i+1,j+1})^2-\frac{24}{\Delta_t}(\phi_{i,j+1}-\phi_{i+1,j})+24\Delta_x\phi_{i,j+\frac{1}{2}}-12\Delta_{xt}p^t_{i,j+1}+12\Delta_x^2 p^t_{i+\frac{1}{2},j}+24(p^x_{i,j}+p^x_{i,j+1})+(p^t(i{+}{+}))^3=0$$

$$\phi_{i+1,j+1}=\phi(i+-)+\Delta_x p^t(i++)+\Delta_x p^t_{i+1,j+1}$$

$$p^x_{i+1,j+1}+p^x_{i+1,j}=p^x_{i,j+1}+p^x_{i,j}+\frac{\Delta_x}{\Delta_t}p^t(i-+)-\frac{\Delta_x}{\Delta_t}p^t(i+1,j+1)+\frac{\Delta_x}{2}\phi(i++)+\frac{\Delta_x}{2}\phi_{i+1,j+1}$$

where:

$$p^t(i\pm+)\equiv p^t_{i,j}\pm p^t_{i,j+1}+p^t_{i+1,j}$$
$$\phi(i++)\equiv \phi_{i,j}+\phi_{i,j+1}-\phi_{i+1,j}$$
$$\Delta_{xt}\equiv\frac{2\Delta_x}{\Delta_t}-\frac{(\Delta_x)^2}{2}$$



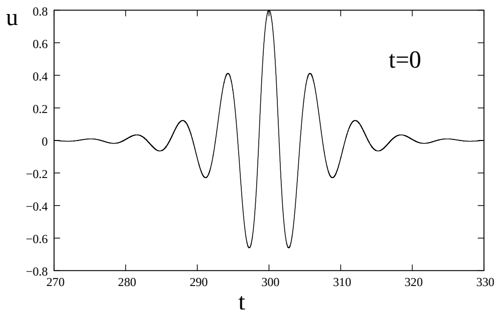
Comparison with other numerical methods (1)

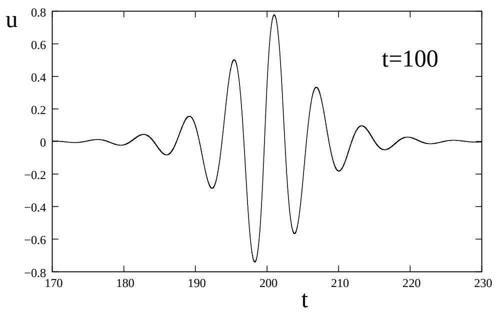
- Split-Step
  - dispersive term: $\frac{\partial}{\partial z}\widehat{U}(z,\omega) = \frac{i}{\omega}\widehat{U}(z,\omega)$ ($\widehat{U}$ - Fourier Transform of U)
  - nonlinear term: $\frac{\partial}{\partial z}U(z,\tau) = 3gU\frac{\partial U}{\partial z}\cdot U$
- Pseudospectral method using Runge-Kutta method

$$\frac{\partial}{\partial z}\widehat{U}(z,\omega) = \frac{i}{\omega}\widehat{U}(z,\omega) - ig\omega\widehat{\{U^3\}}(z,\omega)$$

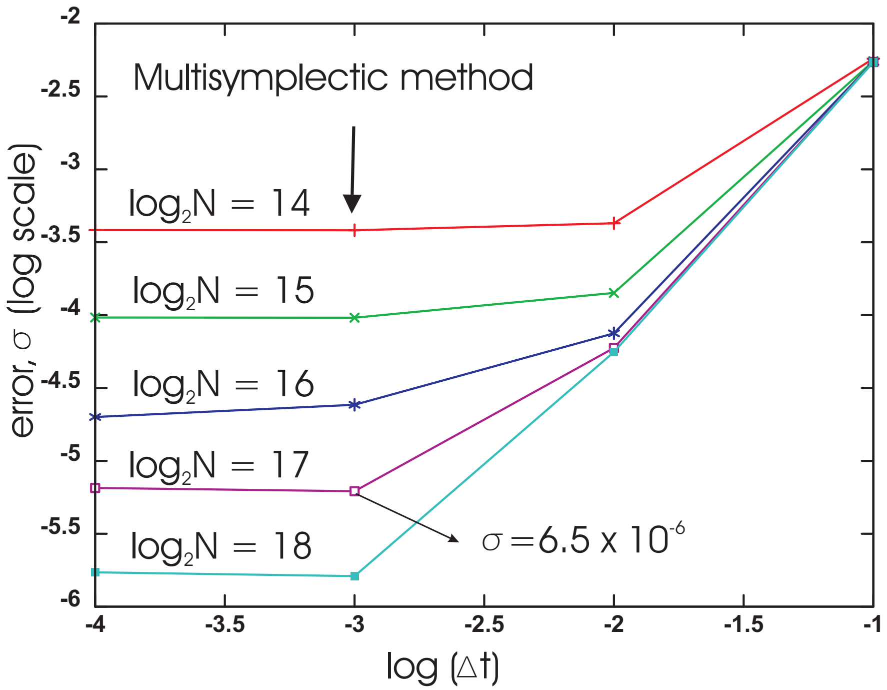
- explicit FD

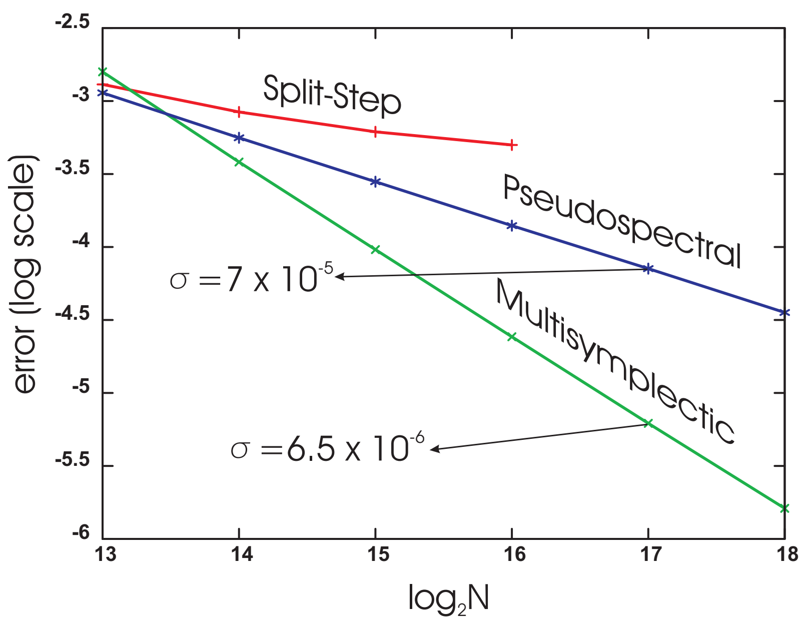
Numerical test:

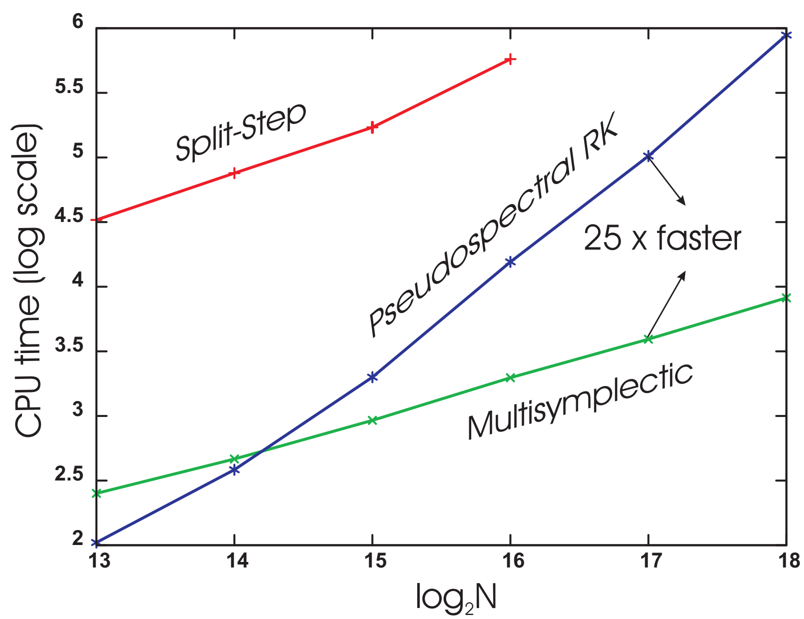




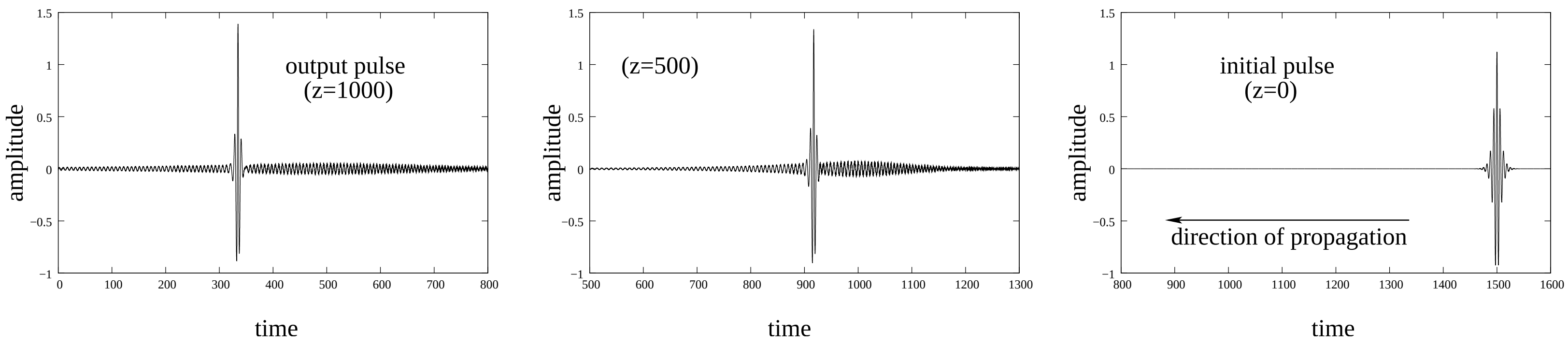
Comparison with other numerical methods (2)

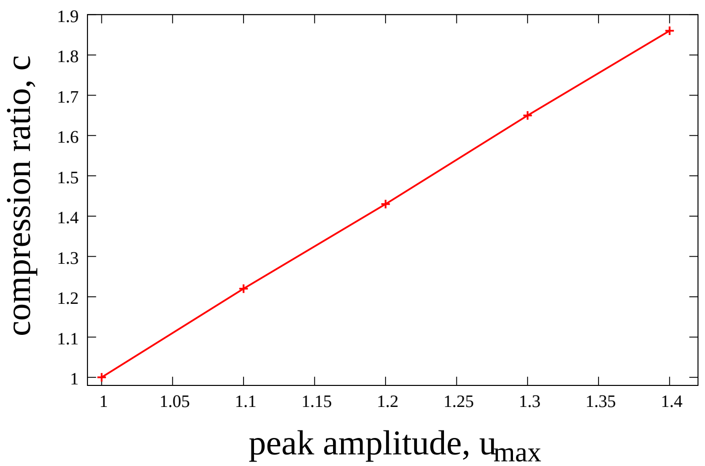


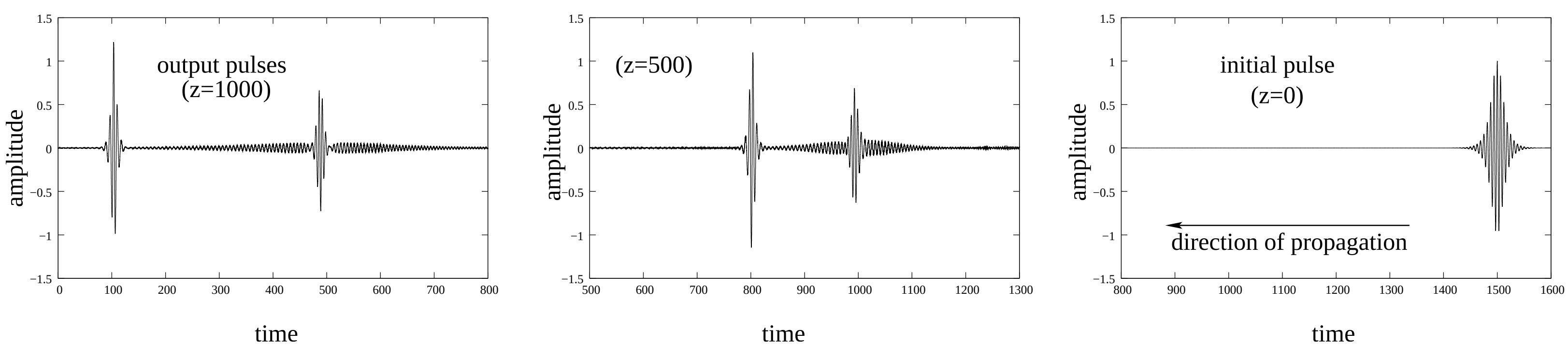




Pulse compression







Conclusions

- Model equations for ultrashort (few-cycle) optical pulses propagating in nonlinear optical fibers $\rightarrow$ Short Pulse Equation (equation for real optical field, no slowly varying envelope approximation)

- SPE $\rightarrow$ NSE (with infinite number of dispersive terms, SPM-term, self-steepening-term, THG-term)

- Soliton solutions of the SPE (Sakovich & Sakovich)

- Effective numerical integrator for the SPE based on the multisymplectic formulation (25 times faster and an order of magnitude more precise than pseudospectral Runge-Kutta method)

- Compression: linear dependence of the compression factor on the peak amplitude